Incoherent matter-wave solitons: Mutual self-trapping of a Bose-Einstein condensate and its surrounding thermal cloud

H. Buljan,[1,2] M. Segev,[1] and A. Vardi[3]
[1] *Physics Department, Technion - Israel Institute of Technology, Haifa 32000, Israel*
[2] *Department of Physics, University of Zagreb, PP 332, Zagreb, Croatia and*
[3] *Department of Chemistry, Ben Gurion University of Negev, Beer Sheva, Israel*

(Dated: February 2, 2008)

We show that a Bose-Einstein condensate and a portion of its surrounding thermal cloud can exhibit mutual self-trapping, supported by the attractive particle interactions, and not by the external confinement. This type of dynamics is characteristic of composite random-phase solitons.

The physics of quantum-degenerate, interacting bose gases closely resembles the behavior of light in nonlinear media. The dynamics of Bose-Einstein condensate (BEC) at zero-temperature is within the Gross-Pitaevskii (GP) mean-field theory described by the nonlinear Schrödinger equation (NLSE) for the condensate order parameter. The same equation describes the evolution of coherent light in nonlinear Kerr medium. This analogy has opened the way for the field of nonlinear atom optics [1, 2] with striking demonstrations of familiar nonlinear optics phenomena such as four wave mixing [3], superradiant Rayleigh scattering [4], and matter-wave amplification [5, 6], carried out with matter-waves. One such phenomenon is the formation of matter-wave solitons [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Experimentally, dark solitons [9, 10] and bright gap solitons [16] were observed in BECs with repulsive interactions, whereas bright solitons [13, 14] were demonstrated in systems with attractive interactions. These experimental results are augmented by extensive theoretical work including predictions on bright [7, 8] and dark [11] matter-wave solitons, lattice solitons [12], and soliton trains [15]. To the best of our knowledge, all previous theoretical efforts on matter-wave solitons have utilized the zero-temperature GP mean-field theory. However, in a realistic system, elementary excitations arising from thermal and/or quantum fluctuations are always present, and the BEC dynamics may be considerably affected by the motion of excited atoms around it (thermal cloud), giving rise to new nonlinear matter-wave phenomena.

Here we present an example of such novel phenomena, and show that a BEC and a portion of its surrounding thermal cloud can exhibit mutually self-trapped motion. This motion is achieved via attractive interactions between particles, and not by the external confinement. We emphasize that the finite-temperature self-trapping produces a truly novel type of matter-wave solitons, where localization is attained not only in spatial density but also in spatial correlations. These self-trapped incoherent matter-waves are analogous to composite random-phase (incoherent) optical solitons [17, 18, 19]; one component is the BEC while the other is a part of its thermal cloud. An important result of this work is that the established analogy between zero-temperature BECs and coherent nonlinear optics can be elevated to the analogy of incoherent light behavior in nonlinear media and BECs at finite-temperatures.

The incoherent matter-wave self-trapping is illustrated within the Hartree-Fock (HF) approximation. First, we solve the static HF problem [20, 21, 22] describing Bose gas with attractive interactions in a confining harmonic potential. We consider the range of (higher) temperatures where excitations are mostly particle-like, and HF solutions are approximately equal to the solutions obtained using the Hartree-Fock-Bogoliubov (HFB) theory in the Popov approximation [21]. We focus on solutions with significant population of the first-excited state. In order to establish mutual-self-trapping, we turn-off the external harmonic confinement, and evolve the BEC and the thermal cloud within the time-dependent HF theory (TDHF) [23, 24, 25]. The signature of mutual self-trapping is a slow separation of the total density profile into two humps, which propagate almost in parallel, and are then pulled back to almost recover the initial density profile. This self-trapped motion is compared to the evolution of precisely the same initial state when both the trap and the interparticle interactions are turned off simultaneously, exhibiting fast matter-wave dispersion.

We consider a system of N interacting bosons placed in a quasi one-dimensional (Q1D) cigar-shaped harmonic potential $V_{ext}(x,y,z) = (\omega_x x^2 + \omega_\perp y^2 + \omega_\perp z^2)/2$, where $\omega_\perp \gg \omega_x$ denote the transverse and the longitudinal frequencies of the trap, respectively. The interparticle interaction is approximated by the Q1D contact potential $V(x_1 - x_2) = g_{1D}\delta(x_1 - x_2)$, where $g_{1D} = -2\hbar^2/ma_{1D}$, $a_{1D} \approx -a_\perp^2/a_{3D}$ is the effective 1D scattering length [26, 27], m is the particle mass, $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the size of the lowest transverse mode, while a_{3D} is the 3D scattering length. At finite temperatures, this system can be described by the Hartree-Fock Bogoliubov (HFB) pairing theory [21]. We are interested in the regime of higher temperatures, where the thermal cloud is sufficiently large; in this limit, anomalous pairings are negligible, the excitations are mostly particle-like, and the HFB formalism can be approximated with the HF theory [21]. In our case, the system is initially at thermal

equilibrium. The density of the condensate $|\langle\hat{\psi}(x,t)\rangle|^2$, and the noncondensed particles $\langle\tilde{\psi}^\dagger(x,t)\tilde{\psi}(x,t)\rangle$ are calculated within the static HF approximation [20, 21, 22]:

$$H_{sp}\phi_0^{(s)} + g_{1D}[n_c^{(s)}(x) + 2n_t^{(s)}(x)]\phi_0^{(s)} = e_0\phi_0^{(s)}(x), \quad (1)$$

$$H_{sp}u_j^{(s)} + g_{1D}[2n_c^{(s)}(x) + 2n_t^{(s)}(x)]u_j^{(s)} = e_j u_j^{(s)}(x). \quad (2)$$

Here, $H_{sp} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_x^2 x^2$; $\phi_0^{(s)}(x)$ is the condensate wavefunction, with eigenvalue e_0; $u_j^{(s)}(x)$ are the wavefunctions of the excited states, with eigenvalues e_j, $j = 1, 2, \ldots$; $n_c^{(s)}(x) = N_c|\phi_0^{(s)}(x)|^2$ is the condensate density, where N_c is the number of condensed particles; $n_t^{(s)}(x) = \sum_j N_j|u_j^{(s)}|^2$ is the density of the excited particles; the number of particles populating the jth excited state, N_j, is determined by the Bose distribution, $N_j = [\exp(\frac{e_j-\mu}{k_B T}) - 1]^{-1}$. The chemical potential μ is set by the constraint $N = N_c + \sum_j N_j$; all wavefunctions are normalized to unity. We solve Eqs. (1) and (2) self-consistently, and use the static solution as the initial condition to study dynamics without confinement.

The dynamics is studied within the TDHF approximation, which involves the coupled evolution of the condensate wavefunction $\phi_0(x,t) = \langle\hat{\psi}(x,t)\rangle$, and the correlation function $\rho(x_1,x_2,t) = \langle\tilde{\psi}^\dagger(x_2,t)\tilde{\psi}(x_1,t)\rangle$ [23, 24, 25]:

$$i\hbar\frac{\partial\phi_0(x,t)}{\partial t} = H_{sp}\phi_0 + g_{1D}[n_c(x,t) + 2n_t(x,t)]\phi_0, \quad (3)$$

$$\begin{aligned} i\hbar\frac{\partial\rho}{\partial t} &= [H_{sp}(1) - H_{sp}(2)]\rho + 2g_{1D}[n_c(x_1,t) + n_t(x_1,t) \\ &- n_c(x_2,t) - n_t(x_2,t)]\rho(x_1,x_2,t), \end{aligned} \quad (4)$$

where $n_c(x,t) = N_c|\phi_0(x,t)|^2$ and $n_t(x,t) = \rho(x,x,t)$. The correlation function ρ is a part of the single particle density matrix $\langle\hat{\psi}^\dagger(x_2,t)\hat{\psi}(x_1,t)\rangle = \phi_0^*(x_2,t)\phi_0(x_1,t) + \rho(x_1,x_2,t)$ corresponding to excitations. Equations of motion (3) and (4) for the condensate wavefunction and the excitations describe the evolution of the BEC and the thermal cloud even outside of equilibrium [23]. Initial conditions at $t = 0$ are $\phi_0(x,t=0) = \phi_0^{(s)}(x)$ and $\rho(x_1,x_2,t=0) = \sum_j N_j u^{*(s)}(x_2)u_j^{(s)}(x_1)$, and the evolution is performed without the external potential. However, rather than to use Eqs. (3) and (4), we shall use a fully equivalent, but numerically more convenient approach. The solution for $\rho(x_1,x_2,t)$ may be constructed from $\rho(x_1,x_2,t) = \sum_j N_j u_j^*(x_2,t)u_j(x_1,t)$, where functions $u_j(x,t)$ evolve according to an infinite set of coupled equations,

$$i\hbar\frac{\partial u_j(x,t)}{\partial t} = H_{sp}u_j + g_{1D}2[n_c(x,t) + n_t(x,t)]u_j, \quad (5)$$

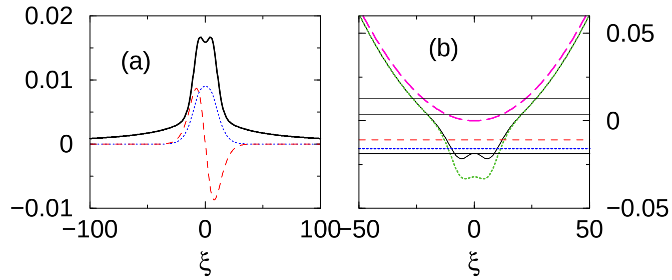

FIG. 1: (color online) Solution of the static Hartree-Fock equations. (a) The total density (solid line), the bell-shaped condensate wavefunction (dotted line), and the dipole-shaped first excited state (dashed line). (b) The (mean-field) potential seen by the condensate (solid line), by the non-condensed particles (dotted line), and the external harmonic potential without the mean field (dashed line). Horizontal lines from the lowest one up depict the chemical potential μ, the condensate eigenvalue e_0, eigenvalue e_1 of the dipole-mode, and eigenvalues e_2, and e_3, expressed in units of $\hbar\omega_\perp$.

where $n_t(x,t) = \sum_j N_j|u_j(x,t)|^2$. Note that when the time-dependence of the wavefunctions is simply $\phi_0(x,t) = \phi^{(s)}(x)\exp(-ie_0 t/\hbar)$ and $u_j(x,t) = u_j^{(s)}(x)\exp(-ie_j t/\hbar)$, the equations of motion (3) and (5) reduce to the static HF equations (1) and (2).

In what follows we present results of a numerical calculation based on the described HF formalism, demonstrating the mutually self-trapped motion of the BEC and a portion of its thermal cloud. The parameters of the calculation are chosen to resemble the experimental parameters of Ref. [13]. We consider $N = 2.2\ 10^4$ ^{7}Li atoms in a harmonic trap with $\omega_\perp = 4907$ Hz ($a_\perp = \sqrt{\hbar/m\omega_\perp} \approx 1.35\ \mu$m), and $\omega_x = 35$ Hz ($a_x = \sqrt{\hbar/m\omega_x} \approx 16.0\ \mu$m). The 3D scattering length $a_{3D} = -3.1\ 10^{-11}$ m corresponds to a nonlinear parameter of $N|a_{3D}| \approx 0.68\ \mu$m, and is tunable by the Feshbach resonance technique [13]. The temperature is $k_B T/\hbar\omega_\perp = 16$. The notable differences from the experiment of Ref. [13] are in the longitudinal trapping frequency, which is an order of magnitude smaller in our calculation, and in the temperature, which is here chosen to produce a sufficiently large thermal cloud and to ensure the validity of the HF formalism.

While the temperature $k_B T$ in our simulation is higher than the transverse level spacing $\hbar\omega_\perp$ whereas a 'true' 1D geometry calls for $k_B T < \hbar\omega_\perp$ [26, 27], the use of a Q1D formalism is still justified because the first $\omega_x/\omega_\perp \sim 140$ states are essentially 1D (they are in the lowest state of the transverse Hamiltonian). As shown below, only a few of the lowest excited states actively participate in the self-trapping process. Therefore, a proper inclusion of the transverse dimension in the calculation would lead to some rescaling of the parameters, but would not influence the self-trapping process observed in our quasi-1D calculation. Moreover, the simulations as well as the experiment of [13], are far from the Tonks-Girardeau regime

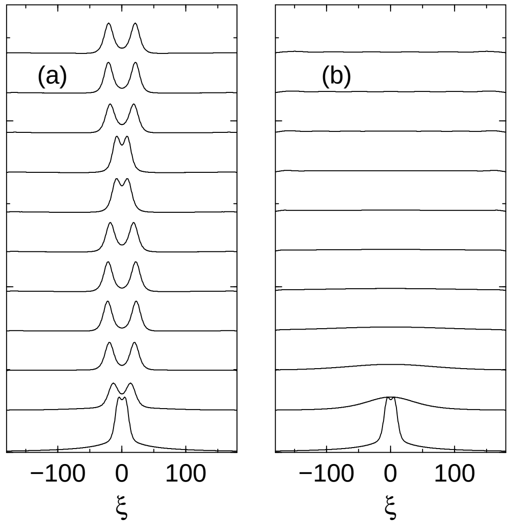

FIG. 2: The evolution of the system (a) with interactions present, and (b) without interactions. In both cases, initial conditions were simply the static HF solution from Fig. 1, while external potential was turned off during evolution. Graphs show the total density profiles at equally spaced intervals from $t = 0$ up to $t = 1.2 \times 2\pi/\omega_x$; the x-coordinate is $\xi = x/a_\perp$.

of impenetrable bosons [26]; our calculations are all in the weak interaction regime $N|a_{1D}|/a_x \sim 10^7 \gg 1$, thus justifying the use of a mean-field approach (we note parenthetically that the weak interaction regime for a 1D gas is attained 'counterintuitively' at *high* densities [26]). The stability of the confined, attractively interacting condensate against collapse [7, 29] was numerically verified by evolving it with random initial noise on top of all equilibrium modes; the stability is underpinned by the use of parameters resembling the experiment [13].

The system is initially in equilibrium. Fig. 1(a) illustrates the total density $n_c^{(s)}(x)+n_t^{(s)}(x)$ of the stationary HF calculation. The total density profile (solid line) is double humped, offering a clear signature of the significant population of the first excited $u_1^{(s)}$ state, which has a dipole-like spatial profile (dashed line). The condensate fraction is $N_c/N \approx 0.24$, while the population of the first excited state is $N_1/N \approx 0.093$. The excitation energies and the mean-field potentials affecting the condensate and the non-condensed cloud are plotted in Fig. 1(b). The eigenvalue e_1 of the first excited state is below zero, along with the condensate eigenvalue e_0 and the chemical potential μ ($\mu \to e_0$ when $T \to 0$). Thus, a large fraction of the thermal cloud $[N_1/(N-N_c) \approx 0.12]$ is self-localized jointly with the condensate, rather than localized by the external harmonic potential. The same conclusion is also inferred from the double-well structure of the mean-field potentials [Fig. 1(b)].

The static evidence for mutual self trapping is confirmed by studying the evolution of the system, once the trapping potential along x is suddenly turned off, as in the experiment of Ref. [13]. The system is suddenly taken out of equilibrium and consequently starts to evolve. We simulate the dynamics by solving Eqs. (3) and (5) with the standard split-step Fourier technique. In the spirit of Ref. [13], we compare the x-unconfined dynamics of the system in the presence of interparticle interactions [Fig. 2(a)] to its time evolution when both the confinement in x and the interactions are turned off [Fig. 2(b)]. In the absence of interactions, we clearly observe a fast dispersion of the total density. After approximately $t_d \sim 1/3 \times 2\pi/\omega_x$, there are hardly any particles left within the window shown in Fig. 2(b). In contrast, when interactions are present, we observe self-trapped motion [Fig. 2(a)]. The two humps begin to separate, because the trapping potential which provided a balance to the kinetic energy term is no longer present. However, due to the attractive particle interactions, the two humps separate very slowly and move almost in parallel [Fig. 2(a)]. Subsequently, the two humps are pulled back, and the initial density roughly recovers its initial appearance. We emphasize that a significant portion of the atoms within the self-trapped entity ($N_1/N_c \sim 0.38$) belong to the thermal cloud, and that they strongly affect the BEC dynamics. Atoms from higher excited states ($j > 10$) disperse quickly even with interactions present, and are essentially spectators of the self-trapped motion.

After the confining potential is turned off, the condensate would be completely depleted after some time-period, which is proportional to the collision time τ_c, because relaxation is attained through collisions. The TDHF formalism is clearly inadequate to depict this depletion since it conserves all populations including the condensate fraction. However, since in the weak interaction regime the collisional time is much longer than the trap period $\tau_c\omega_x \gg 1$ [30], relaxation is slow compared to characteristic trap times. We can therefore safely use the TDHF method to carry out simulations lasting up to several dispersion times $\tau_{disp} \sim 1/3 \times 2\pi/\omega_x \ll \tau_c$ in order to demonstrate self-trapping.

The self-trapped entities presented here represent partially coherent matter waves. Thus, it is important to study the complex degree of coherence of these matter-waves, $\mu(x, x', t) = \rho(x, x', t)/\sqrt{\rho(x, x, t)\rho(x', x', t)}$. Fig. 3 shows the evolution of $|\mu(x, x', t)|$ corresponding to our self-trapped entity. We see that spatial correlation is finite. During the initial stage of the evolution $|\mu(x, x', t)|$ broadens indicating the increase of coherence, but after this initial stage it stays practically unchanged. This corresponds to the fact during the initial stage of the evolution, a portion of the thermal cloud that is not mutually trapped with the BEC disperses, which initially increases the coherence. We emphasize that for zero-temperature GPE solitons, the pair correlation function factorizes as $\rho(x, x') = \phi^*(x)\phi(x')$, which yields $\mu(x, x') = 1$, corresponding to coherent matter-waves. Our incoherent self-

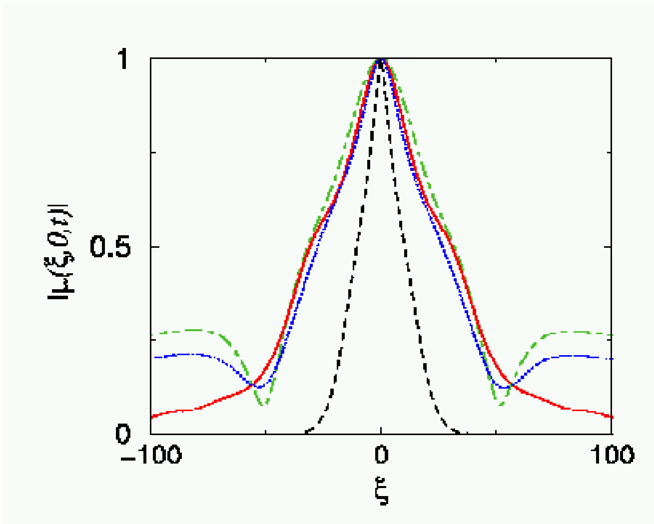

FIG. 3: (color online) The complex degree of coherence $|\mu(x,x',t)|$ of a partially-coherent, self-trapped matter-wave at times $t=0$ (black dashed line), $t=0.4\Delta t$ (red solid line), $t=0.8\Delta t$ (green dot-dashed line), and $t=1.2\Delta t$ (blue dotted line); $\xi=x/a_\perp$, $x'=0$, and $\Delta t=2\pi/\omega_x$.

trapped matter-waves are thus rather special in that they correspond to localization of *entropy* and spatial correlation, as well as to localization of density.

Before closing, we return to the analogy between the propagation of *incoherent* light in nonlinear media, and the behavior of BECs at finite temperatures. Incoherent light can be described by the mutual coherence function $B(x_1,x_2,z) = \langle E^*(x_2,z,t)E(x_1,z,t)\rangle$ [18], where $E(x,z,t)$ is the randomly fluctuating field. The evolution of $B(x_1,x_2,z)$ along the propagation axis z is described by an equation equivalent in structure to Eq. (4) describing the evolution of correlations $\rho(x_1,x_2,t)$ in time (see Ref. [18]). Moreover, the TDHF equations (5) are analogous to the Manakov equations that describe incoherent light in nonlinear media [18, 19]. The analogy is not complete, due to the fact that the Bose wavefunction is affected by a different mean-field than the thermal cloud, while in optics it is usual (but not the rule) that all fields 'see' the same nonlinear change in the index of refraction. Furthermore, in the full HFB approximation, the $U(1)$ symmetry is broken, which yields phenomena not encountered in incoherent nonlinear optics. Nevertheless, we expect that many nonlinear phenomena with incoherent light will find its counterpart in BECs at finite temperatures.

In conclusion, we have demonstrated that a BEC and a portion of its surrounding thermal cloud can exhibit motion analogous to composite random-phase (incoherent) optical solitons. The relation between BEC and nonlinear optics is thus elevated to the analogy between nonlinear incoherent optical waves and nonlinear matter-waves at finite-temperature. The predicted incoherent matter-wave structures represent novel correlation solitons which resemble localized second-sound entropy waves. Work is underway to go beyond HF approximation and study the role of pairing in lower temperature soliton dynamics.

[1] G. Lens, P. Meystre, and E.M. Wright, Phys. Rev. Lett. **71**, 3271 (1993).
[2] S. L. Rolston and W. D. Phillips, Nature **416**, 219 (2002).
[3] L. Deng *et al.*, Nature **398**, 218 (1999).
[4] S. Inouye *et al.*, Science **285**, 571 (1999).
[5] M. Kozuma *et al.*, Science **286**, 2309 (1999).
[6] S. Inouye *et al.*, Nature **402**, 641 (1999).
[7] P.A. Ruprecht *et al.*, Phys. Rev. A **51**, 4704 (1995).
[8] V.M. Perez-Garcia, H. Michinel, and H. Herrero, Phys. Rev. A **57**, 3837 (1998).
[9] S. Burger *et al.*, Phys. Rev. Lett. **83**, 5198 (1999).
[10] J. Denschlag, *et al.*, Science **287**, 97 (2000).
[11] Th. Busch, and J.R. Anglin, Phys. Rev. Lett. **84**, 2298 (2000).
[12] A. Trombettoni and A. Smerzi, Phys. Rev. Lett. **86**, 2353 (2001).
[13] L. Khaykovich *et al.*, Science **296**, 1290 (2002).
[14] K.E. Strecker, *et al.*, Nature **417**, 150 (2002).
[15] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. Lett. **91**, 080405 (2003).
[16] B. Eiermann *et al.*, Phys. Rev. Lett. **92**, 230401 (2004).
[17] M. Mitchell *et al.*, Phys. Rev. Lett. **77**, 490 (1996); M. Mitchell and M. Segev, Nature (London) **387**, 880 (1997).
[18] D.N. Christodoulides *et al.*, Phys. Rev. E **63**, 035601 (2001).
[19] M. Mitchell *et al.*, Phys. Rev. Lett. **79**, 4990 (1997).
[20] V.V. Goldman, I.F. Silvera, and A.J. Legget, Phys. Rev. B **24**, 2870 (1981).
[21] A. Griffin, Phys. Rev. B **53**, 9341 (1996).
[22] M. Houbiers and H.T.C. Stoof, Phys. Rev. A **54**, 5055 (1996); T. Beregman, Phys. Rev. A **55**, 3658 (1997).
[23] N.P. Proukakis and K. Burnett, J. Res. Natl. Inst. Stand. Technol. **101** 457 (1996); N.P. Proukakis, K. Burnett, and H.T.C. Stoof, Phys. Rev. A **57**, 1230 (1998).
[24] A. Minguzzi, and M.P. Tossi, J. Phys.: Condens. Matter **9** 10211 (1997).
[25] V. Chernyak, S. Choi, and S. Mukamel, Phys. Rev. A **67**, 053604 (2001).
[26] V. Dunjko, V. Lorent, and M. Olshanii, Phys. Rev. Lett. **86**, 5413 (2001).
[27] H. Moritz *et al.*, Phys. Rev. Lett. **91**, 250402 (2003).
[28] While strictly speaking the 1D critical temperature is $T_c = 0$, there is significant increase in the population of the lowest energy state below $k_BT_c = \hbar\omega_x N/\ln(2N)$, see W. Ketterle, and N.J. van Drutten, Phys. Rev. **54**, 656 (1996).
[29] Yu. Kagan, A.E. Muryshev, and G. Schlyapnikov, Phys. Rev. Lett. **81**, 933 (1998).
[30] Yu. Kagan, E.L. Surkov, and G.V. Shlyapnikov, Phys. Rev. A **54**, 1753 (1996).